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Geometric scaling of correlation decay in chaotic billiards

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We investigate decay properties of correlation functions in a class of chaotic billiards. We provide numerical evidence that velocity autocorrelation functions decay exponentially, with a rate scaling in a simple way with the (uniform) curvature of the dispersing arcs. Return probabilities, i.e., correlation functions of characteristic functions of subsets of the phase space, appear to follow a slower than exponential decay law.

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In this paper we present some numerical experiments performed on a class of two-dimensional chaotic billiards. In particular, we will be concerned with decay properties of correlation functions: though extensive investigations have been carried out both from a rigorous point of view [1,2], and by numerical methods [3–6], the situation is still somehow controversial, and a considerable effort is still devoted to this problem [7,8].

We will consider diamond (D) billiards; see Fig. 1. The dynamics refers to a point particle with unit velocity bouncing elastically against the boundary; S^t denotes the continuous time evolution, acting on the phase space $\mathcal{M}$, which is the Cartesian product of the set of configuration coordinates with the unit circle parametrized by the angle formed by $\vec{v}$ with a fixed direction. The system is ergodic and mixing [9] and the invariant measure is proportional to the Lebesgue measure on $\mathcal{M}$, $d\mu(z)=(2\pi A)^{-1}\, dx\, dy\, d\omega$ (A being the area of the billiard region).

The mixing property guarantees that correlation functions vanish asymptotically. The goal here is to characterize their decay; as correlation functions are intimately linked to transport coefficients this is an issue of the utmost physical import. For a function $f(z)$ defined on the phase space, the correlation function (CF) is defined as

$$C_f(t) = \int_{\mathcal{M}} d\mu(z)\, f(S^t z) f(z) - \left(\int_{\mathcal{M}} d\mu(z) f(z) \right)^2. \tag{1}$$

Generally speaking, in the presence of the K-system property the asymptotic behavior of $C_f(t)$ at large t depends on the choice of the function f, very slow decay laws being possible if very irregular functions f are allowed. From a physical point of view, a natural choice for f is a component of the velocity, which is directly related to diffusive properties of the system. With this choice, and provided arcs do not meet tangentially, the systems we consider are believed to satisfy a bound [2] (originally established for finite horizon Lorentz gas [1,10]), in terms of a stretched exponential

$$|C_f(t)| \leq \exp(\gamma t^{\eta}) \tag{2}$$

with $1/2 \leq \eta \leq 1$. A major problem in the rigorous analysis of diamonds consists in Markov partitions not being finite [11,10]; this makes it hard to prove exponential decay, even if the system is purely hyperbolic.

However, for a class of somehow simpler systems (piecewise linear automorphisms on the two-torus), Chernov [12] proved pure exponential decay of a class of CFs in the presence of both hyperbolicity and singularities (that induce problems in constructing Markov partitions). Quite recently, a technique has been proposed [8,13] to prove exponential decay without making use of Markov partitions; though not directly applicable to the systems under investigation, it might provide an approach to the rigorous analysis of billiards. For D systems some numerical experiments are consistent with a pure exponential decay of the velocity CFs [7] (see also [6]). Earlier investigations supported subexponential decay for a different type of CF, namely, that of the characteristic function of a subset of the phase space [5], which, properly normalized, has the meaning of a return probability to the chosen subset.

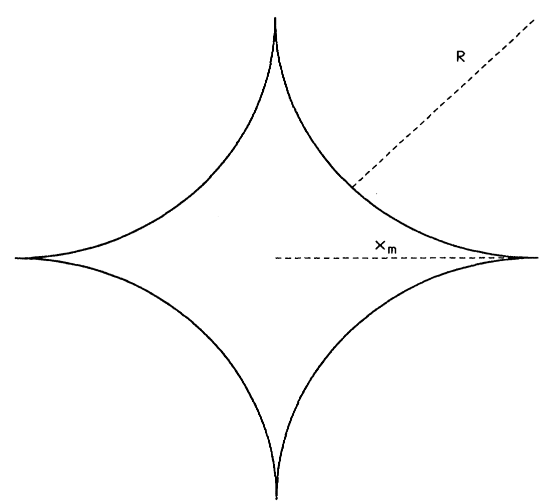

FIG. 1. Boundary of the D system: R is the radius of curvature of the arcs, x_m is taken equal to 1 in simulations.

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We summarize our main results: we observe pure exponential decay for velocity-velocity correlation functions; the corresponding decay rates scale according to a power law with R (radius of the dispersing arcs). By considering other types of correlations we observe different behaviors: crossing statistics exhibit exponential decay (with rates differing from the former ones), whereas for the CFs of characteristic functions of some subset of the phase space the observed decay is even qualitatively different, slower than exponential.

We start by investigating velocity-velocity correlation functions; we consider components of the velocity along a direction which is diagonal with respect to the orientation of Fig. 1 (that is along the direction indicated by R). We have numerically evaluated these functions by Monte Carlo integration over a set of $N_{\rm ph}$ initial conditions. The random generator we used in performing the integration is based on a subtractive method (suggested by Knuth [14]); occasionally we checked that linear congruential methods did not alter the results; as a further check we reproduced the results of [7] with our methods.

For the D "standard" [5,7] parameter value ($R=2.236\dots$) we reobtained the data shown by [7] for $C(t)=\langle v_{\sharp}(t)v_{\sharp}(0)\rangle$: for a number of initial conditions $N_{\rm ph}=10^7$ we get $\gamma_v=0.56\pm0.02$ (which is fully consistent with the result of [7] once a difference in the overall length scale is properly taken into account). As before, γ_v indicates the exponential decay rate [$\mathcal{C}(t)\sim e^{-\gamma_v t}$].

The main difficulty in numerical investigations is that when decay is very fast statistical errors become relevant after a very short time. In particular, the error involved in estimating phase averages through Monte Carlo integration scales like $N^{-1/2}$ (N being the number of sampling points) [15]. In the standard case the decay is so rapid that obtaining an affordable longer time sequence is hopeless; we then investigated a number of other D cases in which the value of R is increased (R ranging from 1.58 to 27.59). The structure of $\mathcal{C}(t)$ is always the same: it exhibits pure exponential decay and a superimposed oscillatory behavior (there is a regular alternation of maxima and minima); the periods of oscillations are quite close to the period $2\sqrt{2}$ of the oscillations of the CF in the limit case $R\mapsto\infty$ [6]. Some of the results are plotted in Figs. 2 and 3.

To better appreciate the limits of validity of our simulations, some comments are in order. In all cases, the locations of maxima are well within the bound imposed by statistical error (whose order of magnitude is, however, estimated heuristically [15]. Other potential sources of errors [7] involve an estimate of the initial transient and errors implied by exponential divergence of trajectories. In particular, in [7] it has been remarked that after a time $T_{\rm max}$ [such that $\varepsilon \geqslant e^{\lambda T_{\rm max}} d_m$ where d_m is the machine precision (10^{-16} for double precision calculations) and ε is the statistical error] the phase averages employed in correlation function calculations are to be interpreted as *hydrodynamical* averages, as we cannot any more claim that we are following real trajectories of the system. This is a subtle issue and we have no theoretical breakthrough (like extension of shadowing properties) to put forward: empirically it is, however, true that D systems seem robust with respect to this issue, as we checked by rerunning our simulations with lower precision, and breaking the $T_{\rm max}$ barrier does not produce any sensible change in the estimate of slopes. Nevertheless, in the absence of reliable theoretical warrants, we restricted our plots (and γ_v estimates) to simulations within the $T_{\rm max}$ limit (in the case of Fig. 3 $T_{\rm max}\approx 93.1$).

In Fig. 4 we show how γ_v varies with R: for large values of R a power-law behavior of the exponent seems to show up: we do not have any scaling argument capable of explaining this behavior if only approximately, but we believe that this is worth further investigation (maybe via some mean-field treatment, in the same spirit as [16]). The same figure

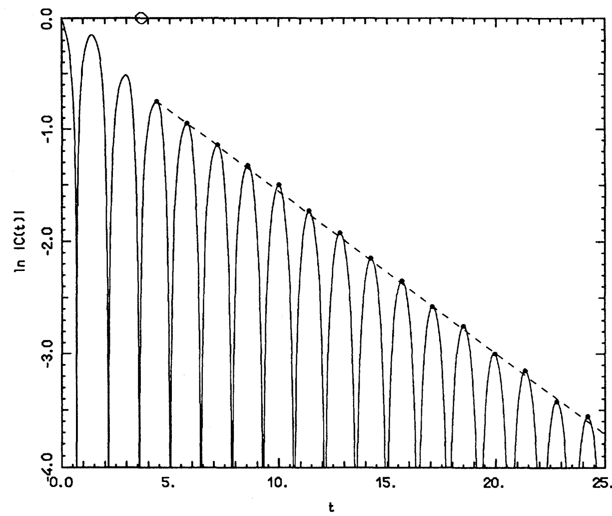

FIG. 2. $\ln|C(t)|$ vs t for a D case ($R=5$, $N_{\rm ph}=5\times10^6$), statistical errors relative to maxima are within circles: the dashed line is a least square fit using maximum points. The circle gives a rough estimate of the transient (Lyapunov) time $\tau\approx 2/\lambda$.

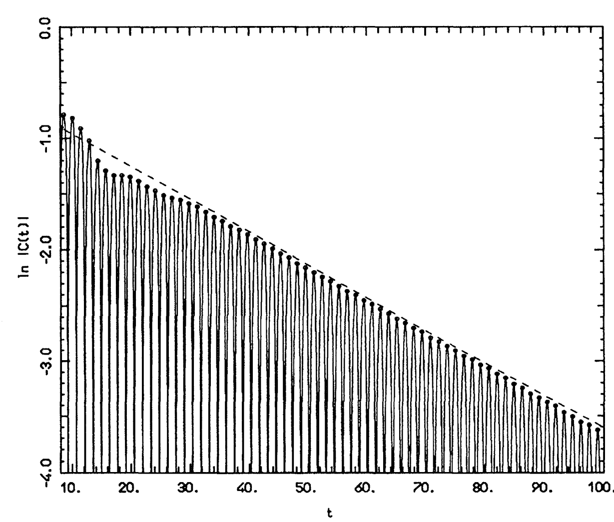

FIG. 3. $\ln|C(t)|$ vs t for a D case ($R=14.866$, $N_{\rm ph}=10^7$): the dashed line is a least square fit using maximum points, statistical errors at maxima are within circles. The estimated transient time is shorter than the starting time in the plot.

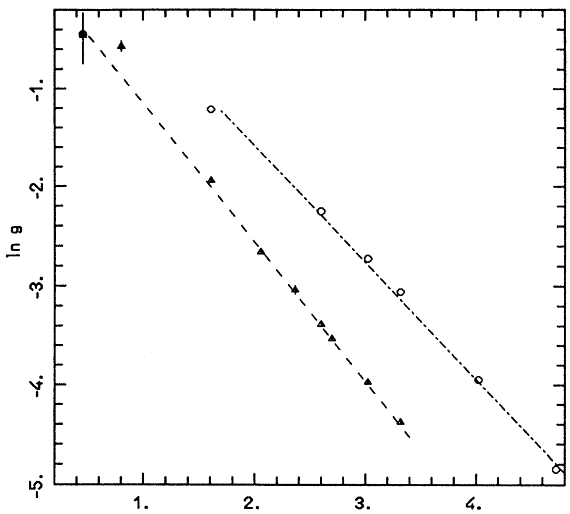

FIG. 4. $\ln \gamma_v$ (△) (errors marked by vertical bars) and $\ln \gamma_\tau$ (errors within symbol heights) (○) vs $\ln R$: the dashed line has a slope -1.41, while the dashed-dotted line has a slope -1.18.

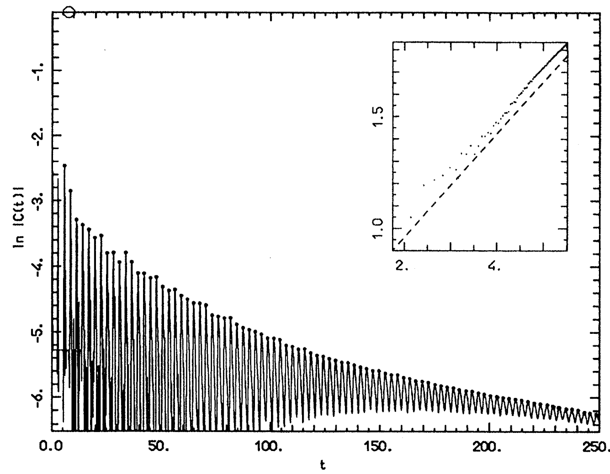

FIG. 5. $\ln \mathscr{C}_{\mathscr{A}}(t)$ vs t for a D case ($R=20.518$, $N_{\rm ph}=2\times 10^7$, $x\in[0,0.25]$ $y\in[0,0.25]$, $\theta_v\in[0.1,1.1]$); the inset represents the maxima (indicated by ○ in the main graph) but plotted as $\ln|\ln C_{\max}(t)|$ vs $\ln t$: a stretched exponential fit on these points gives $\gamma_{\rm SE}=0.23$. The circle gives a rough estimate of the transient (Lyapunov) time $\tau\approx 2/\lambda$.

also reports on other correlation decay rates, related to "crossing" statistics [17,18], in the following sense: we cut the configuration space of the system under investigation, along the shortest segment joining two facing arcs and run a single (or a sample of) trajectory, recording the sequence of times $\{t_\alpha\}$ from one crossing to the next. From this data set we approximately reconstruct $\wp(t)$ (probability distribution functions for the crossing time); then the probability that the particle has not crossed the boundary within time t is built:

$$P_{\rm int}(t) = \int_t^\infty \wp(\tau)\, d\tau. \quad (3)$$

These are monotonically decreasing functions of their argument and they are normalized to 1 at the origin.

Our interest in these quantities originates from the fact that their integral plays a role analogous to correlation functions: it has been argued by Karney [17] (see also [18]) that the following quantity

$$C_\tau = \int_\tau^\infty dz\, P_{\rm int}(z) \quad (4)$$

is proportional to the probability that a particle does not cross the boundary during a time interval of size τ; an analogous reasoning has been invoked by [19] in dealing with the transition to chaos in standardlike mappings. In our simulations C_τ is observed to decay exponentially; Fig. 4 reports how γ_τ varies with R ($C_\tau \sim e^{-\gamma_\tau \tau}$).

According to earlier suggestions [5], we finally consider correlation functions involving characteristic functions of some subset $\mathscr{A}$ of the phase space (we denote such a function by $\chi_{\mathscr{A}}$); if $\mathscr{A}\subset\mathscr{M}$ the corresponding correlation function is defined as

$$\mathscr{C}_{\mathscr{A}}(t) = \left| \frac{\langle \chi_{\mathscr{A}}(t)\chi_{\mathscr{A}}(0)\rangle_{\mathscr{A}} - \mathrm{vol}(\mathscr{A})}{1-\mathrm{vol}(\mathscr{A})} \right|, \quad (5)$$

where $\langle\ \rangle_{\mathscr{A}}$ denotes a phase average, ruled by the invariant measure, with initial conditions belonging to $\mathscr{A}$. By inspecting Fig. 5 we see that some of the regularity features we reported upon are lost: the oscillations look less regular than in the previous case (and this is connected to symmetry loss in the choice of $\mathscr{A}$), but, more strikingly, no clear pure exponential decay is exhibited. The pattern of maxima is instead compatible with a subexponential behavior; see Fig. 5. Fitting this pattern with a stretched exponential yields $\gamma\sim 0.23$ with an error of a few percent. To the extent to which numerical results can be held indicative of the true asymptotic behavior, the observed difference in the behavior of velocity CFs and of characteristic function CFs may be traced back to the different degree of regularity of the two types of functions.

Though the problem of correlation decay has deserved the attention of the dynamical system community for a long time, many issues are still unresolved and much effort is still devoted to gaining a better understanding. In the present paper we have addressed the problem of numerical investigations on dynamical systems with singularities. Velocity-velocity correlation functions are shown to exhibit pure exponential decay; moreover, the decay rates seem to scale regularly with variations of the geometrical parameter; similar scaling relations appear to hold for survival probabilities. This exponential decay matches recent rigorous results [12,8] for other hyperbolic systems with singularities, and other numerical experiments [7] as well. Correlation functions involving less smooth phase functions exhibit a more complex behavior: although this is not in contrast with the current theoretical understanding, a precise assessment of the connection between smoothness and decay laws (as given, e.g., in [20] for toral automorphisms) for this class of systems is still an unresolved issue, which we believe to be relevant to a deeper understanding of typical behavior in dynamical systems.

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